Note on exponents associated with Y-systems

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Abstract

Let (X_n, ℓ) be the pair consisting of the Dynkin diagram of finite type X_n and a positive integer $\ell \geq 2$, called level. Associated to the pair (X_n, ℓ), we obtain the Y-system, which is the set of algebraic relations, and the Dynkin quiver $Q(X_n, \ell)$. On $Q(X_n, \ell)$, we have a structure of the cluster algebra. By their periodicity, we are able to define a sequence of integers called exponents. Mizuno provide the conjectural formulas for the exponents in his research. We will present the some results for Mizuno's conjecture. Furthermore, we will see the relationship with the theory of cluster algebras and the representation theory of affine Lie algebras.

1 Introduction

Cluster algebra is an algebraic structure introduced by Fomin and Zelevinsky in the early 2000's. The cluster algebras is generated by the quiver $\mathcal{Q}$, which is a directed graph consisting of vertices and arrows, and variables $y = (y_1, \ldots, y_{|\mathbf{I}|})$ called cluster and coefficients $Y = (Y_1, \ldots, Y_{|\mathbf{I}|})$. A defining feature of cluster algebras is that they generate new algebraic structure thorough the operation called mutation. By such structure, cluster algebras play an important role in various fields such as combinatorics, representation theory, geometry, and even physics. Denote the set of vertices of $\mathcal{Q}$ by $\mathbf{I}$. In our research, we assume that quivers have no 1,2-cycles. There exists a triple $\gamma = (\mathcal{Q}, \mathbf{m}, \nu)$ called mutation loop on the cluster algebra, where $\mathbf{m}$ is a subsequence of $\mathbf{I}$ and ν is a permutation on $\mathbf{I}$. By the mutation loop γ, we can obtain certain special operation μ_γ called cluster transformation. Let X_n be a finite type Dynkin diagram and ℓ be a positive integer such that $\ell \geq 2$. Inoue, Iyama, Keller, Kuniba and Nakanishi constructed the quiver $Q(X_n, \ell)$ with the mutation loop $\gamma(X_n, \ell)$ such that the corresponding cluster transformation μ_γ has a periodicity in [3, 4]. The periodicity of the cluster transformation μ_γ means the following identity holds.

$$\mu_\gamma^{T(\ell + h^\vee)}(y) = y,$$

where $h^\vee$ is the dual Coxeter number and

$$T = \begin{cases} 1 & (X = A,\ D,\ E), \\ 2 & (X = B,\ C,\ F), \\ 3 & (X = G). \end{cases}$$

A Y-system is a system of algebraic relation determined from the root system of type X_n and the positive integer $\ell \geq 2$ called level. It is known that the (restricted constant) Y-system has a unique positive real solution. Consider the fixed point equation $\mu_\gamma(y) = y$. Then we find that the fixed point equation has a unique positive real solution and it is written in terms

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of the solution of the Y-system. Denote such solution by η. By the periodicity of the cluster transformation μ_γ, we obtain the matrix

$$J_\gamma(\eta) = \left(\frac{\partial \mu_i(y)}{\partial y_j} \right)\bigg|_{y=\eta} \qquad \text{such that} \qquad J(\eta)^{T(\ell+h^\vee)} = I,$$

where $\mu_i(y)$ stands for the i-th component of $\mu_\gamma(y)$. Therefore all eigenvalues of $J_\gamma(\eta)$ are written by

$$e^{\frac{2\pi i m_1}{T(\ell+h^\vee)}}, e^{\frac{2\pi i m_2}{T(\ell+h^\vee)}}, \dots, e^{\frac{2\pi i m_{|\mathbf{I}|}}{T(\ell+h^\vee)}}$$

by using a sequence of nonnegative integers such that $0 \leq m_1 \leq m_2 \leq \cdots \leq m_{|\mathbf{I}|} < T(\ell + h^\vee)$. We call this sequence $\mathcal{E} = (m_1, m_2, \dots, m_{|\mathbf{I}|})$ as the exponents on $Q(X_n, \ell)$, and associated with level ℓ Y-system of type X_n.

Mizuno provided the conjectural formula for exponents on $Q(X_n, \ell)$ by using the root system of type X_n. Let Δ be the root system of type X_n. We have the following two decompositions.

$$\Delta = \Delta_+ \cup \Delta_-, \quad \Delta = \Delta_{\text{long}} \cup \Delta_{\text{short}},$$

where Δ_+, Δ_-, Δ_{long} and Δ_{short} are the sets of positive roots, negative roots, long roots and short roots respectively. Set ρ as the half sum of positive roots. Denote a nondegenerate invariant symmetric bilinear from on Δ by $\langle \cdot, \cdot \rangle$. This form is fixed so that $\langle \alpha, \alpha \rangle = 2$ if $\alpha \in \Delta_{\text{long}}$. Associated to the pair (X_n, ℓ), we define two polynomials

$$N_{X_n,\ell}(z) = \prod_{i=1}^{n} \frac{z^{T(\ell+h^\vee)} - 1}{z^{T/t_i} - 1}, \quad D_{X_n,\ell}(z) = D^{\text{long}}_{X_n,\ell}(z) D^{\text{short}}_{X_n,\ell}(z),$$

where $t_i = 2/\langle \alpha_i, \alpha_i \rangle$, and the polynomials $D^{\text{long}}_{X_n,\ell}(z)$ and $D^{\text{short}}_{X_n,\ell}(z)$ are defined by

$$D^{\text{long}}_{X_n,\ell}(z) = \prod_{\alpha \in \Delta_{\text{long}}} \left(z^T - e^{\frac{2\pi i \langle \rho, \alpha \rangle}{\ell + h^\vee}} \right), \quad D^{\text{short}}_{X_n,\ell}(z) = \prod_{\alpha \in \Delta_{\text{short}}} \left(z - e^{\frac{2\pi i \langle \rho, \alpha \rangle}{\ell + h^\vee}} \right).$$

Then we have the following conjecture

Conjecture 1.1. *[6, Conjecture 3.8] Let X_n be a finite type Dynkin diagram, and ℓ be a positive integer such that $\ell \geq 2$. Let $\gamma = \gamma(X_n, \ell)$ be the mutation loop on the quiver $Q(X_n, \ell)$. Then the following identity holds for the characteristic polynomial of $J_\gamma(\eta)$.*

$$\det(zI - J_\gamma(\eta)) = \frac{N_{X_n,\ell}(z)}{D_{X_n,\ell}(z)}.$$

The consistency of Conjecture 1.1 follows from the coincident of asymptotics of partition q-series on T-data and normalized q-characters of certain weight subspace of modules of the affine Lie algebra. In order to verify such relationship, we reveiw each aspects of the theory of cluster algebras and the representation theory of affine Lie algebras.

First, consider a triple (B_+, B_-, D) of matrices, whose entries are Laurent polynomials with integer coefficients. The triple (B_+, B_-, D) is a T-datum if $B_\pm$ can be written as $B_\pm = N_0 - N_\pm$ by a triple of matrices (N_0, N_+, N_-) satisfying certain conditions. For the pair of the affine Lie algebra of type $X_N^{(r)}$ and the integer $\ell \geq 2$, we are able to construct certain T-datum $\beta(X_N^{(r)}, \ell)$, which induce the mutation loop associated with Cartan matrix.

Using the T-datum $\beta(X_N^{(r)},\ell)$, we are able to define geometric series called total partition q-series denoted by $\mathcal{Z}_\beta(q)$. Using the Rogers dilogarithm function, we obtain certain positive real number a. Then we have the following asymptotics.

$$\lim_{\epsilon\to 0}\mathcal{Z}_\beta(e^{-\epsilon})e^{-\frac{a}{\epsilon}} = \sqrt{\frac{\det(B_+)}{\det(I-J_\gamma(\eta))}},$$

where $\epsilon \to 0$ means the limit in the positive real axis.

Next, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type X_n equipped with a Dynkin automorphism σ of order r'. Denote its Cartan subalgebra by $\mathfrak{h}$. Let $R=\bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ be a root lattice and $P=\bigoplus_{i=1}^n \mathbb{Z}\varpi_i$ be a weight lattice of $\mathfrak{g}$, where α_i is a simple root and ϖ_i is the fundamental weight respectively. Set $Q=\bigoplus_{i=1}^m \mathbb{Z}\alpha_i \subset R$. Denote the invariant subspace of Q under the action of σ by $Q_{(0)}$ and its dual space by $Q^*_{(0)}$. Correspond to the choice of the pair (X_N,σ), the affine Lie algebra of type $X_N^{(r)}$ is realized as the canonical central extension of the loop algebra $L\mathfrak{g}$. As the vector space, we have $\tilde{\mathfrak{g}} = L\mathfrak{g}\oplus\mathbb{C}c\oplus\mathbb{C}d$, where c is the center and d is the degree operator. Note that $Q_{(0)}$ should be understood as the root lattice of $\bar{\mathfrak{g}}$, where $\bar{\mathfrak{g}}$ is the subalgebra of $\tilde{\mathfrak{g}}$ obtained by removing 0-th node of Dynkin diagram of $\tilde{\mathfrak{g}}$.

Let $L(\ell\Lambda_0)$ be the highest weight $\tilde{\mathfrak{g}}$-module with highest weight $\ell\Lambda_0$. Since $L(\ell\Lambda_0)$ is integrable highest weight module, we have canonical decomposition $L(\ell\Lambda_0)=\bigoplus_{n\geq 0}L(\ell\Lambda_0)_n$ so that $\dim(L(\ell\Lambda_0)_n)<\infty$. Therefore the following function is well-defined

$$\chi_\ell(\tau) = q^{\frac{c(\ell)}{24r}}\operatorname{tr}_{L(\ell\Lambda_0)}q^{-d},$$

where

$$c(\ell) = \frac{\ell\cdot\dim(\mathfrak{g})}{\ell+h^\vee}.$$

This is the q-character of $L(\ell\Lambda_0)$. By letting $q=e^{2\pi i\tau}$, the function $\chi_\ell(\tau)$ converges to a holomorphic function on the upper half plane $\mathbb{H}=\{\tau\in\mathbb{C}\mid \operatorname{Im}(\tau)>0\}$. We have the following asymptotics.

$$\lim_{\tau\downarrow 0}\chi_\ell(\tau)e^{-\frac{\pi i c(\ell)}{12r\tau}} = \sqrt{\left|Q^*_{(0)}/(\ell+h^\vee)\right|}\prod_{\alpha\in\Delta_+}2\sin\frac{\pi\langle\rho,\alpha\rangle}{\ell+h^\vee},$$

where $\tau\downarrow 0$ is the limit in the positive imaginary axis. Furthermore, the RHS is calculated via the value of

$$\lim_{z\to 1}\frac{N_{X_n,\ell}(z)}{D_{X_n,\ell}(z)}.$$

The notion of the parafermionic space more clearly connects these asymptotics. Set $Q(\ell)=\bigoplus_{i=1}^n\mathbb{Z}t_i\ell\alpha_i$ as the sublattice of Q. We define the parafermionic space of highest weight $\ell\Lambda_0$ such that $\ell\geq 2$ by

$$L(\ell\Lambda_0)_{Q(k)} = \bigoplus_\mu\left(L(\ell\Lambda_0),\mu\right),$$

where μ takes value in $\ell\Lambda_0+Q/Q(\ell)$ and

$$(L(\ell\Lambda_0),\mu)=\{v\in L(\ell\Lambda_0)\mid (h\otimes t^m)v=\delta_{m,0}\mu(h)v\quad (h\in\mathfrak{h},\ m\geq 0)\}$$

is the weight subspace isomorphic to the commutant of $L(\ell\Lambda_0)$. Then we consider the following function

$$b_\mu^{(\ell)}(\tau) = q^{-\frac{c(\ell)-n}{24r}}\operatorname{tr}_{(L(\ell\Lambda_0),\mu)}q^{-d_V-d},$$

where

$$d_V|_{(L(\ell\Lambda_0),\mu)} = \frac{\langle\mu_{(0)},\mu_{(0)}\rangle}{2\ell}$$

and $\mu_{(0)}$ is the projection of μ onto $Q_{(0)}$. This is the branching function for the pair $(\mathfrak{g}, \mathfrak{h})$. By the definition of the parafermionic space, we find that

$$\mathrm{ch}\ L(\ell\Lambda_0)_{Q(\ell)} = q^{\frac{c(\ell)-n}{24r}} \sum_{\mu} b_{\mu}^{(\ell)}(q) \prod_{i=1}^{n} y_i^{m_i},$$

where $\mu = \ell\Lambda_0 + m_1\alpha_1 + \cdots + m_n\alpha_n$ such that $0 \leq m_i \leq t_i\ell - 1$.

Then we have the following conjecture.

Conjecture 1.2. *[2, Conjecture 5.3], [6, Conjecture 6.2] The fermionic formula of partition q-series associated with T-data $\beta(X_N^{(r)}, \ell)$ such that $\ell \geq 2$ is given by the q-character of parafermionic space $L(k\Lambda_0)_{Q(k)}$. Namely, we have*

$$q^{-\frac{c(\ell)-n}{24}} \mathcal{Z}_\beta(q) = \sum_{\substack{1\leq i\leq n \\ 0\leq m_i\leq t_i\ell-1}} b^{(\ell)}_{\ell\Lambda_0+m_1\alpha_1+\cdots+m_n\alpha_n}(q).$$

Now, we consider the T-datum β associated with the pair $(X_n^{(1)}, \ell)$. Under the assumption that Conjecture 1.2 is true, we have the following identity .

$$\lim_{\epsilon\to 0} \mathcal{Z}_\beta(e^{-\epsilon})e^{-\frac{a}{\epsilon}} \left(\lim_{z\to 1} \frac{N_{X_n,\ell}(z)}{D_{X_n,\ell}(z)}\right)^{-\frac{1}{2}}.$$

These suggest the existence of some relationship between the theory of cluster algebras and the representation theory of affine Lie algebras.

2 Main Results

For Conjecture 1.1, Mizuno and the author presented several results. By using the explicit positive real solution of restricted constant Y-system, we are able to construct vectors ψ such that

$$J_\gamma(\eta)\psi = \lambda\psi$$

for certain roots of unity λ. Namely, we will find all eigenvalues of matrix $J_\gamma(\eta)$. As a consequence, we have the following theorem.

Theorem 2.1. *[6, Theorem 3.9], [9, Theorem 4.3, 4.6] The Conjecture 1.1 holds for $(X_n, \ell) = (A_1, \ell)$, $(A_n, 2)$, $(B_n, 2)$ and $(D_n, 2)$.*

Furthermore, for fermionic conjectural formulas of partition q-series, Butorac, Kožić, Primc, Okado and the author presented the proof of Conjecture 1.2. By vertex algebraic construction, we are able to construct canonical combinatorial bases of highest weight module $L(\ell\Lambda_0)$ and its parafermionic space $L(\ell\Lambda_0)_{Q(\ell)}$. Using such bases, we derive the fermionic character formulas. As a consequence, we have the following theorem.

Theorem 2.2. *[1, Theorem 3.6], [7, Theorem 13], [8, Theorem 17] The Conjecture 1.2 is true for partition q-series associated with T-data $\beta(X_N^{(r)}, \ell)$ such that $\ell \geq 2$.*

Therefore, Theorem 2.2 reveals that Conjecture 1.1 holds for $z = 1$. Furthermore, Theorem 2.2 implies that we are able to propose the problem of calculating the asymptotic of the partition q-series associated with $\beta(X_N^{(r)}, \ell)$ such that $r > 2$ as a future works.

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